NOETHER DECOMPOSITION FOR BIRATIONAL MAPS

by

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Abstract. — Let ϕ be a birational map of the complex projective plane. We know that ϕ can be written as a composition of automorphisms of $\mathbb{P}^2_\mathbb{C}$ and the standard quadratic birational map σ. This writing, that is non-unique, is minimal if the number $\mathfrak{n}(\phi)$ of σ is as small as possible. We prove that if ϕ is of degree $d\geq 2$, then $\left[\frac{\ln d}{\ln 2}\right]\leq\mathfrak{n}(\phi)\leq 2(2d-1)$.

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1. Introduction

A *rational map* ϕ of $\mathbb{P}^2_\mathbb{C}$ is a map of the following type

$$\phi\colon\mathbb{P}^2_\mathbb{C}\dashrightarrow\mathbb{P}^2_\mathbb{C},\qquad (x:y:z)\dashrightarrow\big(\phi_0(x,y,z):\phi_1(x,y,z):\phi_2(x,y,z)\big)$$

where the ϕ_i's are homogeneous polynomials of the same degree, and without common factor. The *degree* of ϕ is the degree of the ϕ_i's. A *birational map* ϕ of $\mathbb{P}^2_\mathbb{C}$ is a rational map of $\mathbb{P}^2_\mathbb{C}$ for which there exists a rational map ψ of $\mathbb{P}^2_\mathbb{C}$ such that $\phi\psi=\psi\phi=\mathrm{id}$.

Examples 1.1. —
- if $d=1$ then ϕ is a birational map given by linear forms, *i.e.* ϕ is an element of $\mathrm{Aut}(\mathbb{P}^2_\mathbb{C})=\mathrm{PGL}(3;\mathbb{C})$.
- in the case $d=2$ we have the following examples:

  $$\sigma\colon (x:y:z)\dashrightarrow(yz:xz:xy),\qquad \rho\colon (x:y:z)\dashrightarrow(xy:z^2:yz),\qquad \tau\colon (x:y:z)\dashrightarrow(x^2:xy:y^2-xz).$$

  As we will see these three maps play an important role in the description of the set of quadratic birational maps of $\mathbb{P}^2_\mathbb{C}$.

If ϕ denotes a birational map of the complex projective plane, we denote by $\mathscr{O}(\phi)$ the orbit of ϕ under the action of $\mathrm{PGL}(3;\mathbb{C})\times\mathrm{PGL}(3;\mathbb{C})$

$$\mathscr{O}(\phi)=\big\{A_1\phi A_2\,|\,A_1,A_2\in\mathrm{PGL}(3;\mathbb{C})\big\}.$$

***Theorem 1.2* ([3])**. — *Let ϕ be a birational map of $\mathbb{P}^2_\mathbb{C}$ of degree 2, then ϕ belongs to*

$$\mathscr{O}(\sigma)\cup\mathscr{O}(\rho)\cup\mathscr{O}(\tau).$$

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The birational maps of $\mathbb{P}^2_{\mathbb{C}}$ form a group called *Cremona group* and denoted by $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$. The oldest statement concerning this group is the following:

***Theorem 1.3* ([2]).** — *The group* $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ *is generated by* $\mathrm{Aut}(\mathbb{P}^2_{\mathbb{C}})$ *and* σ.

In other words any birational map ϕ of $\mathbb{P}^2_{\mathbb{C}}$ can be written

$$(A_1)\sigma A_2\sigma\ldots\sigma A_{n-1}\sigma(A_n)$$

with A_i in $\mathrm{PGL}(3;\mathbb{C})$. This writing is of course non-unique, for example

$$\sigma=\sigma(2x:y:z/2)\sigma(2x:y:z/2)\sigma.$$

We will say that the writing of ϕ is *minimal* if the number of σ in this writing is as small as possible, and we will denote by $\mathfrak{n}(\phi)$ this number.

***Examples 1.4*.** —

- If A denotes an automorphism of $\mathbb{P}^2_{\mathbb{C}}$, then $\mathfrak{n}(A)=0$.
- One has

$$\mathfrak{n}(\sigma)=1,\qquad \mathfrak{n}(\rho)=2,\qquad \mathfrak{n}(\tau)=4.$$

So, according to Theorem 1.3, if $\phi\in\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is of degree 2, then $\mathfrak{n}(\phi)\leq 4$.

The question is: if $\phi\in\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ is of degree d, can we bound $\mathfrak{n}(\phi)$?

***Theorem A*.** — *Let ϕ be a birational map of $\mathbb{P}^2_{\mathbb{C}}$ of degree $d\geq 2$ then*

$$\left[\frac{\ln d}{\ln 2}\right]\leq\mathfrak{n}(\phi)\leq 2(2d-1).$$

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2. A first bound

We will first of all give a lower but "immediate" bound.

Let ϕ be an element of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$; it is given by

$$(x:y:z)\dashrightarrow(\phi_0(x,y,z):\phi_1(x,y,z):\phi_2(x,y,z))$$

for some homogeneous polynomials of the degree d, and without common factor. The *linear system* of ϕ is the preimage of the linear system of lines of $\mathbb{P}^2_{\mathbb{C}}$ and is denoted Λ_ϕ. The degree of the curves of Λ_ϕ is obviously d.

If ϕ has some points where ϕ is not defined, we choose one, that we denote $p_1\in\mathbb{P}^2_{\mathbb{C}}$; let $\pi_1\colon S_1\to\mathbb{P}^2_{\mathbb{C}}$ be its blow-up. The map $\psi_1=\phi\pi_1\colon S_1\dashrightarrow\mathbb{P}^2_{\mathbb{C}}$ is a birational one. If ψ_1 has at least one base point, we again choose one that we denote $p_2\in S_2$ and $\pi_2\colon S_2\to S_1$ its blow-up. Again the map $\psi_2=\psi_1\pi_2\colon S_2\dashrightarrow\mathbb{P}^2_{\mathbb{C}}$ is a birational one. We continue the process until ψ_n becomes a morphism. Let us justify the existence of such a n. The linear system Λ_ϕ consists of curves of degree d all passing through the p_i's with multiplicity m_i. Recall that a blow-up $\pi\colon Y\to X$ of a point $p\in X$ induces the map $\pi^*\colon\mathrm{Pic}(X)\to\mathrm{Pic}(Y)$, which sends a curve $\mathcal{C}\subset X$ into $\pi^{-1}(\mathcal{C})$. Furthermore if $\mathcal{C}\subset X$ is an irreducible curve, the strict transform of $\mathcal{C}$, denoted $\widetilde{\mathcal{C}}$, is obtained by taking the closure of $\pi^{-1}(\mathcal{C}\setminus\{p\})$. In $\mathrm{Pic}(Y)$ we have

$$\pi^*(\mathcal{C})=\widetilde{\mathcal{C}}+m_p(\mathcal{C})E$$

where $E = \pi^{-1}(p)$. Applying this n times the members of Λ_{ψ_n} are equivalent to

$$dL - \sum_{i=1}^{n} m_i E_i$$

so these curves have self-intersection $d^2 - \sum_{i=1}^{n} m_i^2$ that has to be non-negative; therefore the number n exists.

$$\begin{tikzcd}
S_n \arrow[d, "\pi_n"'] \arrow[ddddr, "\psi_n"] & \\
\vdots \arrow[d] & \\
S_2 \arrow[d, "\pi_2"'] \arrow[ddr, dashed, "\psi_2"] & \\
S_1 \arrow[d, "\pi_1"'] \arrow[dr, dashed, "\psi_1"] & \\
\mathbb{P}^2_{\mathbb{C}} \arrow[r, dashed, "\phi"'] & \mathbb{P}^2_{\mathbb{C}},
\end{tikzcd}$$

Denote by $\mathcal{E}_i \subset S_i$ the (-1)-curve $\pi_i^{-1}(p_i)$ and by

$$E_i = (\pi_{i+1} \dots \pi_n)^* \mathcal{E}_i \in \mathrm{Pic}(S_n).$$

The points p_i are called *base points* of ϕ, some of them belong to $\mathbb{P}^2_{\mathbb{C}}$ (these are the common zeros of the ϕ_i often called the indeterminacy points of ϕ) some of them don't; we say that these last one are *infinitely near* $\mathbb{P}^2_{\mathbb{C}}$. We denote by $\mathrm{Base}\,\phi$ the set of base points of ϕ.

By construction the map ψ_n is a birational morphism, let us denote it ξ. In fact any birational morphism between smooth projective surfaces is a sequence of blow-ups so

$$\xi = \xi_q \dots \xi_1$$

where ξ_i is the blow-up of a point $q_i \in S'_i$ with $S'_0 = \mathbb{P}^2_{\mathbb{C}}$ and $S'_q = S_n$ (it follows from the computations of the rank of the Picard group that $q = n$).

The linear system Λ_ϕ of ϕ corresponds to the strict pull-back by ξ of the system $\mathcal{O}_{\mathbb{P}^2}(1)$. Let L be a general line, which does not pass through the p_i; its pull-back $\xi^{-1}(L)$ corresponds to a smooth curve on S_n which has self-intersection 1 and genus 0. By adjunction formula one gets

$$(\xi^{-1}(L))^2 = 1, \qquad \xi^{-1}(L) \cdot K_{S_n} = -3.$$

Since the members of Λ_ξ are equivalent to $dL - \sum_{i=1}^{n} m_i E_i$ and since $K_{S_n} = -3L + \sum_{i=1}^{n} E_i$ one has

$$\left\{ \begin{array}{l} d^2 - \sum_{i=1}^{n} m_i^2 = 1 \\ 3d - \sum_{i=1}^{n} m_i = 3 \end{array} \right. \tag{2.1}$$

Remark that not all solutions of (2.1) correspond to the base points of a birational map of degree d.

Nevertheless the solution $m_0 = d-1$, $m_1 = \ldots = m_{2d-2} = 1$ is realized, for example by (*see* [**4**])

$$f_d = (xz^{d-1} + y^d : yz^{d-1} : z^d);$$

more precisely f_d has one indeterminacy point with multiplicity $d-1$ and $2d-2$ base points infinitely near each of them having multiplicity 1. Applying the previous construction one gets that f_d is the composition of two sequences of $(2d-1)$ blow-ups, that is $2(2d-1)$ blow-ups. A blow-up belongs to $\mathscr{O}(\rho)$ so we need 2 σ to write it. Then $\mathfrak{n}(f_d) \leq 4(2d-1)$.

Lemma 2.1. — *Let ϕ be a birational map of $\mathbb{P}^2_{\mathbb{C}}$ of degree $d \geq 2$, then ϕ has at most $2d-1$ base points.*

Proof. — Let us denote by $p_1, \ldots, p_n$ the base points of ϕ and by m_i the multiplicity of p_i.

The inequality $d \geq m_i + 1$ holds for any i. In fact if p_i belongs to $\mathbb{P}^2_{\mathbb{C}}$, the pencil of lines passing through p_i has to intersect positively the linear system so $d \geq m_i + 1$. If p_j is infinitely near to a point p_i we have $m_j \leq m_i$ so $d \geq m_i + 1 \geq m_j + 1$.

Let us order the p_i's such that $m_1 \geq m_2 \geq \ldots \geq m_n$; the following inequality holds

$$m_1 + m_2 + m_3 \geq d. \tag{2.2}$$

Indeed, from (2.1) we get

$$\sum_{i=1}^n m_i^2 - m_3 \sum_{i=1}^n m_i = d^2 - 1 - 3(d-1)m_3$$

that gives

$$(d-1)\big(m_1 + m_2 + m_3 - (d+1)\big) = (m_1 - m_3)\big(d - (1+m_1)\big) + (m_2 - m_3)\big(d - (1+m_2)\big) + \sum_{i=4}^n m_i(m_i - m_3)$$

Of course $\displaystyle\sum_{i=4}^n m_i(m_i - m_3) \geq 0$ and as we just see $d \geq m_i + 1$ for any i; hence

$$m_1 + m_2 + m_3 - (d+1) \geq 0.$$

$\square$

We have established the following statement.

Proposition 2.2. — *Let ϕ be a birational map of degree $d \geq 2$; then*

$$\mathfrak{n}(\phi) \leq 4(2d-1).$$

3. A bound for birational maps of $\mathbb{P}^2_{\mathbb{C}}$ coming from polynomial automorphisms of $\mathbb{C}^2$

This section is based on [**6**]; in this article the author gives a geometric proof of JUNG's Theorem:

***Theorem 3.1* ([5])**. — *The group of polynomial automorphisms of $\mathbb{C}^2$ denoted $\mathrm{Aut}(\mathbb{C}^2)$ has a structure of amalgamated product:*

$$\mathrm{Aut}(\mathbb{C}^2) = \mathrm{A} *_{\mathrm{S}} \mathrm{E}$$

where

$$\mathrm{A} = \{(x,y) \mapsto (a_1x + b_1y + c_1, a_2x + b_2y + c_2) \,|\, a_1b_2 - a_2b_1 \neq 0\},$$
$$\mathrm{E} = \{(\alpha x + P(y), \beta y + \gamma) \,|\, \alpha, \beta, \gamma \in \mathbb{C}, \alpha\beta \neq 0, P \in \mathbb{C}[y]\}$$

and $\mathrm{S} = \mathrm{A} \cap \mathrm{E}$.

Before giving the sketch of the proof, let us recall what are HIRZEBRUCH surfaces: a HIRZEBRUCH *surface*, denoted by $\mathbb{F}_n$, is a ruled surface over the projective line defined by

$$\mathbb{F}_n = \mathbb{P}_{\mathbb{P}^1_\mathbb{C}}(\mathcal{O}_{\mathbb{P}^1_\mathbb{C}}) \oplus \mathcal{O}_{\mathbb{P}^1_\mathbb{C}}(n) \qquad \forall n \geq 2.$$

The surface $\mathbb{F}_0$ is isomorphic to $\mathbb{P}^1_\mathbb{C} \times \mathbb{P}^1_\mathbb{C}$ and $\mathbb{F}_1$ is isomorphic to $\mathbb{P}^2$ blown up at a point. HIRZEBRUCH surfaces for $n > 0$ have a special rational curve $s_\infty(\mathbb{F}_n)$ on them: $\mathbb{F}_n$ is the projective bundle of $\mathcal{O}(-n)$ and the curve $s_\infty(\mathbb{F}_n)$ is the zero section. This curve has self-intersection $-n$, and this is the only irreducible curve with negative self-intersection. The only irreducible curves with zero self-intersection are the fibers $f_\infty(\mathbb{F}_n)$ of $\mathbb{F}_n$.

We say that a birational map $\phi\colon S \dashrightarrow \mathbb{P}^2_\mathbb{C}$ from a surface S to $\mathbb{P}^2_\mathbb{C}$ *comes from a polynomial automorphism of* $\mathbb{C}^2$ if

1. $S = \mathbb{C}^2 \cup D$ where D is a union of irreducible curves called divisor at infinity;
2. $\mathbb{P}^2_\mathbb{C} = \mathbb{C}^2 \cup L$ where L is a line;
3. ϕ induces an isomorphism between $S \smallsetminus D$ and $\mathbb{P}^2_\mathbb{C} \smallsetminus L$.

This situation implies strong constraints on the base points of ϕ:

***Lemma 3.2* ([6], Lemma 9).** — *Let $\phi\colon S \dashrightarrow \mathbb{P}^2_\mathbb{C}$ be a birational map from a surface S to $\mathbb{P}^2_\mathbb{C}$ that comes from a polynomial automorphism of $\mathbb{C}^2$. Then*

(i) *ϕ has only one base point in $\mathbb{P}^2_\mathbb{C}$ on the divisor at infinity of S;*

(ii) *ϕ has base points $p_1, \ldots, p_n$, with $n \geq 1$, such that*
- *p_1 is the indeterminacy point in $\mathbb{P}^2_\mathbb{C}$;*
- *for any $i = 2, \ldots, n$ the point p_i belongs to the exceptional divisor obtained by blowing up p_{i-1};*

(iii) *any of the irreducible curves contained in the divisor at infinity of S is contracted onto a point by ϕ;*

(iv) *the first curve contracted by π_2 is the strict transform of a curve contained in the divisor at infinity of S;*

(v) *in particular if $S = \mathbb{P}^2_\mathbb{C}$ then the first curve contracted by π_2 is the strict transform of the line at infinity.*

Let us explain the strategy used by LAMY to prove JUNG's Theorem. Let ϕ be a birational map of $\mathbb{P}^2_\mathbb{C}$ coming from a polynomial automorphism of $\mathbb{C}^2$ of degree n.

The first step is the blow up the only indeterminacy point of ϕ; one thus gets the following

$$\begin{tikzcd} & \mathbb{F}_1 \arrow[dr, dashed, "\phi_1"] & \\ \mathbb{P}^2_\mathbb{C} \arrow[ur, "\varphi_1"] \arrow[rr, dashed, "\phi"'] & & \mathbb{P}^2_\mathbb{C}, \end{tikzcd}$$

where φ_1^{-1} is the blow up to $(1:0:0)$, and $\#\mathrm{Base}\,\phi_1 = \#\mathrm{Base}\,\phi - 1$. According to Lemma 4.1 the only indeterminacy point of ϕ_1 is $f_\infty(\mathbb{F}_1) \cap s_\infty(\mathbb{F}_1)$.

The second step is based on the following statement:

***Lemma 3.3*.** — *Let $k \geq 1$, and let $\psi\colon \mathbb{F}_k \dashrightarrow \mathbb{P}^2_\mathbb{C}$ be a birational map that comes from a polynomial automorphism of $\mathbb{C}^2$. Assume that the unique proper indeterminacy point of ψ is the point $p = s_\infty(\mathbb{F}_k) \cap f_\infty(\mathbb{F}_k)$. Let*

us consider the following commutative diagram

$$\begin{tikzcd} & \mathbb{F}_{k+1} \arrow[dr, dashed, "\psi'"] & \\ \mathbb{F}_k \arrow[ur, "\varphi"] \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^2_{\mathbb{C}}, \end{tikzcd}$$

where φ *is the blow up of the point* p *composed with the contraction of the strict transform of* $f_\infty(\mathbb{F}_k)$*. Then the birational map* $\psi' = \psi\varphi^{-1}$ *satisfies the following properties:*

1. $\#\mathrm{Base}\,\psi' = \#\mathrm{Base}\,\psi - 1$*;*
2. *the indeterminacy point of* ψ' *is on* $f_\infty(\mathbb{F}_{k+1})$*.*

After the first step we are under the assumptions of Lemma 3.3 with $k = 1$. We get a map $\psi' \colon \mathbb{F}_2 \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ with a unique indeterminacy point; this point lies on $f_\infty(\mathbb{F}_2)$. Repeating this process as soon as we satisfy assumptions of Lemma 3.3 one gets the following diagram

$$\begin{tikzcd} & \mathbb{F}_n \arrow[dr, dashed, "\phi_2"] & \\ \mathbb{F}_1 \arrow[ur, dashed, "\varphi_2"] \arrow[rr, dashed, "\phi_1"'] & & \mathbb{P}^2_{\mathbb{C}}, \end{tikzcd}$$

where φ_2 is obtained by applying $n-1$ times Lemma 3.3. Furthermore one has

$$\#\mathrm{Base}\,\phi_2 = \#\mathrm{Base}\,\phi_1 - n + 1$$

and the indeterminacy point of ϕ_2 is on $f_\infty(\mathbb{F}_n)$ but not on $s_\infty(\mathbb{F}_n)$.

The third step relies on the following result:

Lemma 3.4. — *Let* $k \geq 2$*, and let* $\psi \colon \mathbb{F}_k \dashrightarrow \mathbb{P}^2_{\mathbb{C}}$ *be a birational map that comes from a polynomial automorphism of* $\mathbb{C}^2$*. Assume that the unique indeterminacy point* p *of* ψ *lies on* $f_\infty(\mathbb{F}_k)$ *but not on* $s_\infty(\mathbb{F}_k)$*. Let us consider the following diagram*

$$\begin{tikzcd} & \mathbb{F}_{k-1} \arrow[dr, dashed, "\psi'"] & \\ \mathbb{F}_k \arrow[ur, "\varphi"] \arrow[rr, dashed, "\psi"'] & & \mathbb{P}^2_{\mathbb{C}}, \end{tikzcd}$$

where φ *is the blow up of* p *composed with the contraction of the strict transform of* $f_\infty(\mathbb{F}_n)$*. Then the map* ψ' *satisfies the two following properties*

1. $\#\mathrm{Base}\,\psi' = \#\mathrm{Base}\,\psi - 1$*;*
2. *the proper indeterminacy point of* ψ' *lies on* $f_\infty(\mathbb{F}_{k-1})$ *but not on* $s_\infty(\mathbb{F}_{k-1})$*.*

After the second step the assumptions of Lemma 3.4 are satisfied. Moreover as soon as $k \geq 3$, the map ψ' given by Lemma 3.4 still satisfies the assumption of this lemma. So after applying $n-1$ times Lemma 3.4 we get

$$\begin{tikzcd} & \mathbb{F}_1 \arrow[dr, dashed, "\phi_3"] & \\ \mathbb{F}_n \arrow[ur, dashed, "\varphi_3"] \arrow[rr, dashed, "\phi_2"'] & & \mathbb{P}^2_{\mathbb{C}}, \end{tikzcd}$$

with $\#\mathrm{Base}\,\phi_3 = \#\mathrm{Base}\,\phi_2 - n + 1$, and the only proper indeterminacy point of ϕ_3 lies on $f_\infty(\mathbb{F}_1)$ but not on $s_\infty(\mathbb{F}_1)$. According to Lemma 4.1 and ZARISKI's Theorem we get

$$\begin{tikzcd} & \mathbb{P}^2_{\mathbb{C}} \arrow[dr, dashed, "\phi_4"] & \\ \mathbb{F}_1 \arrow[ur, "\varphi_4"] \arrow[rr, dashed, "\phi_3"'] & & \mathbb{P}^2_{\mathbb{C}}, \end{tikzcd}$$

where φ_4 is the blow up of some point q whose exceptional divisor is $s_\infty(\mathbb{F}_1)$. Since ϕ_4 is defined up to isomorphism one can assume that $q = (1:0:0)$. Furthermore $\#\mathrm{Base}\,\phi_3 = \#\mathrm{Base}\,\phi_4$.

Conclusion: finally

$$\begin{tikzcd} & \mathbb{P}^2_{\mathbb{C}} \arrow[dr, dashed, "\phi_4"] & \\ \mathbb{F}_1 \arrow[ur, dashed, "\varphi_4\varphi_3\varphi_2\varphi_1"] \arrow[rr, dashed, "\phi"'] & & \mathbb{P}^2_{\mathbb{C}}, \end{tikzcd}$$

where $\varphi = \varphi_4\varphi_3\varphi_2\varphi_1$ is an element of $\mathrm{Aut}(\mathbb{C}^2)$ that preserves the pencil of lines through $(1:0:0)$, *i.e.* $\varphi \in \mathrm{E}$, and $\#\mathrm{Base}\,\phi_4 = \#\mathrm{Base}\,\phi - 2n + 1$.

Hence a birational map ϕ of $\mathbb{P}^2_{\mathbb{C}}$ of degree d that comes from a polynomial automorphism of $\mathbb{C}^2$ can be written as follows

$$\phi = \varphi\psi$$

where ψ is an affine automorphism, φ is a sequence of $2d-1$ blow-ups. Since a blow-up can be written with $2\ \sigma$, the map ϕ can be written $2(2d-1)\ \sigma$.

Remark 3.5. — For $d = 2$, we need $4\ \sigma$: the map $\tau = (x^2 : xy : y^2 - xz)$ can be written $\ell_1\sigma\ell_2\sigma\ell_3\sigma\ell_2\sigma\ell_4$ with

$$\ell_1 = (y-x : 2y-x : z-y+x), \qquad \ell_3 = (-y : x+z-3y : x)$$
$$\ell_2 = (x+z : x : y), \qquad \ell_4 = (y-x : z-2x : 2x-y),$$

and our bound gives $6\ \sigma$.

For $d = 3$, we need $8\ \sigma$: the map $\psi = (xz^2 + y^3 : yz^2 : z^3)$ can be written $\ell_1\sigma\ell_2\sigma\ell_3\sigma\ell_4\sigma\ell_5\sigma\ell_4\sigma\ell_6\sigma\ell_2\sigma\ell_7$ with

$$\ell_1 = (z-y : y : y-x), \qquad \ell_2 = (y+z : z : x), \qquad \ell_3 = (-z : -y : x-y)$$
$$\ell_4 = (x+z : x : y), \qquad \ell_5 = (-y : x-3y+z : x), \qquad \ell_6 = (-x : -y-z : x+y)$$
$$\ell_7 = (x+y : z-y : y),$$

and our bound gives $10\ \sigma$.

4. Proof of NOETHER Theorem and consequences

4.1. NOETHER Theorem ([1]). — An element of $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ which preserves a pencil of rational curves is classically called JONQUIÈRES *transformation*. If ϕ is a JONQUIÈRES map of degree d, it has a base point of multiplicity $d-1$ and $2d-2$ base points of multiplicity 1.

Let ϕ be a birational map of $\mathbb{P}^2_{\mathbb{C}}$ of degree d. Let $p_1, \ldots, p_n$ denote the base points of ϕ and m_i the multiplicity of p_i. Assume that the p_i are ordered such that $m_1 \geq m_2 \geq \ldots \geq m_n$. Set $j_\phi = \frac{d-m_1}{2}$ and let h_ϕ be

the number of p_i, with $i\neq 1$, such that $m_i>j_\phi$. The integer $2j_\phi$ measures the complexity of Λ_ϕ. Remark that $j_\phi\geq\frac{1}{2}$ with equality if and only if ϕ is a JONQUIÈRES transformation.

Lemma 4.1 **([1])**. — *If ϕ is a birational map of $\mathbb{P}^2_{\mathbb{C}}$ of degree $d\geq 2$, the integer h_ϕ satisfies the following properties:*

1. $h_\phi\geq 2$*;*
2. *if* $h_\phi\geq 3$*, then* $\displaystyle\sum_{i=1}^{h} m_i>d$*;*
3. *if* $h_\phi\geq 3$ *and if the points* $p_1,\ldots,p_{h_\phi}$ *are in* $\mathbb{P}^2_{\mathbb{C}}$*, then they are not all aligned.*

Proof. — To see that $h_\phi\geq 2$, it is sufficient to prove that the following inequality holds: $\displaystyle\sum_{i=1}^{h_\phi}(m_i-j_\phi)>m_0-j_\phi$. By definition of h_ϕ we have

$$\sum_{i=1}^{h_\phi} m_i(m_i-j_\phi)\geq\sum_{i=1}^{n} m_i(m_i-j_\phi).$$

But

$$\begin{aligned}\sum_{i=1}^{n} m_i(m_i-j_\phi) &= \sum_{i=1}^{n} m_i^2-j_\phi\sum_{i=1}^{n} m_i\\ &= (d-1)(d-3j_\phi+1)\\ &= d(d-3j_\phi)+3j_\phi-1\\ &= d(m_1-j_\phi)+3j_\phi-1\end{aligned}$$

i.e. $\displaystyle\sum_{i=1}^{n} m_i(m_i-j_\phi)>2j(m_1-j_\phi)$. But for any i the integer m_i is smaller than $2j_\phi$ hence $\displaystyle\sum_{i=1}^{h_\phi} m_i-j_\phi>m_1-j_\phi$.

From $\displaystyle\sum_{i=1}^{h_\phi} m_i-j_\phi>m_1-j_\phi$ one gets $\displaystyle\sum_{i=1}^{h_\phi} m_i>hj_\phi+m_1-j_\phi$. But $h_\phi j_\phi+m_1-j_\phi=d+j(h_\phi-3)$ therefore

$$\sum_{i=1}^{h_\phi} m_i>d+j_\phi(h_\phi-3).$$

As a consequence $\displaystyle\sum_{i=1}^{h_\phi} m_i>d$ as soon as $h_\phi\geq 3$. □

Let q be a quadratic birational map whose indeterminacy points are p_1, A and B, we also say that q is a *quadratic birational map centered at* p_1, A and B. Set $\phi'=\phi q$ and $d'=\deg\phi'$. The idea is the following: choose A and B such that $(j_\phi,h_\phi)>(j_{\phi'},h_{\phi'})$ for the lexicographic order. After a finite number of such steps, one obtains an automorphism of $\mathbb{P}^2_{\mathbb{C}}$.

Let us first assume that p_1 is not the point of largest multiplicity of ϕ'. If Λ and Λ' are two linear systems of $\mathbb{P}^2_{\mathbb{C}}$, the *free intersection* of Λ and Λ' is a non-negative integer, which counts the number of free points, that is points which are not base points of Λ, Λ' in the intersection of a general member of Λ and a general member of Λ'.

The free intersection of a generic line through p_1 and ϕ^*L, where L denotes a pencil of lines of $\mathbb{P}^2_\mathbb{C}$, is $d-m_1=2j_\phi$. If p_1 is a base point of multiplicity m'_1 for ϕ' we have $d'-m'_1=d-m_1=2j_\phi$. If P denotes the base point of largest multiplicity m_P for ϕ' then

$$2j_{\phi'}=d'-m_P<d'-m'_1=2j_\phi.$$

In other words if there exist A and B two points in $\mathbb{P}^2_\mathbb{C}$ such that after composed ϕ with a quadratic birational map centered at A, B and p_1 the point p_1 is not of largest multiplicity then $j_{\phi'}<j_\phi$.

Suppose now that p_1 is the point of largest multiplicity of ϕ'. The point p_1 is the point of largest multiplicity of ϕ'. As we just see, then $j_\phi=j_{\phi'}$. One of the following holds:

$\mathfrak{a}$) there are two points of indeterminacy p_2 and p_3 with multiplicity $m_2>j_\phi$ and $m_3>j_\phi$;

$\mathfrak{b}$) there is at most one indeterminacy point with multiplicity $>j_\phi$ and no base point infinitely near p_1;

$\mathfrak{c}$) there is at most one indeterminacy point with multiplicity $>j_\phi$ and at least one base point infinitely near p_1.

Let us consider all these cases.

$\mathfrak{a}$) Let us consider the quadratic birational map q centered at p_1, p_2 and p_3 and set $\phi'=\phi q$. The multiplicity m'_2 of p_2 for ϕ' is equal to the number of free points of an element of ϕ^*L and the line through p_1 and p_3. By Bezout one has

$$d=m_1+m_3+m'_2.$$

As $d-m_1=m_3+m'_2$ and $d-m_1=2j_\phi$ one has $j_\phi>m'_2$. Similarly $j_\phi>m'_3$. Thus $h_{\phi'}=h_\phi-2$.

$\mathfrak{b}$) Assume that the base points p_1, p_2 and p_3 of ϕ satisfy the following conditions:

- p_1 is of largest multiplicity,
- p_2 belongs to $\mathbb{P}^2_\mathbb{C}$,
- p_3 is infinitely near p_2.

Let us choose a point P in $\mathbb{P}^2_\mathbb{C}$ such that the line through P and p_1 (resp. m and p_2) does not contain base point of ϕ distinct from p_1 (resp. p_2). Let us compose ϕ with the quadratic birational map centered at p_1, p_2 and P. The point P (resp. p_1) becomes a base point of multiplicity $<j_\phi$ (resp. of multiplicity $2j_\phi$) and p_2 an indeterminacy point of multiplicity $m_2>j_\phi$. Hence h is constant and there is one more indeterminacy point with multiplicity $>j_\phi$. Iterating this process we can assume that all the base points of multiplicity $>j_\phi$ are in $\mathbb{P}^2_\mathbb{C}$. Since $h_\phi\geq 2$ we thus are in case $\mathfrak{a}$).

$\mathfrak{c}$) Let A and B be two generic points of $\mathbb{P}^2_\mathbb{C}$. After having composed ϕ with the quadratic birational map centered at A, B and p_1, the integer h has increased by 2 as A and B are of multiplicities $2j_\phi$; in particular $h_\phi\geq 4$. Nevertheless there is no more base point infinitely near p_1. According to $\mathfrak{b}$) we can assume that $p_1,\ldots,p_{h_\phi}$ are in $\mathbb{P}^2_\mathbb{C}$. These points are not all aligned (Lemma 4.1); hence we can apply $\mathfrak{a}$) at least two times and h_ϕ decreases by 4. Finally h_ϕ has decreased by 2.

After repeating $\mathfrak{a}$) a finite number of times, we obtain a map ψ such that $h_\psi<2$ then from Lemma 4.1 we get either $\psi\in\mathrm{Aut}(\mathbb{P}^2_\mathbb{C})$ or $j_\psi<j_\phi$.

4.2. Proof of Theorem A. — The configuration that needs the most σ is the configuration of birational maps coming from polynomial automorphisms of $\mathbb{C}^2$ (*see* §4.1).

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